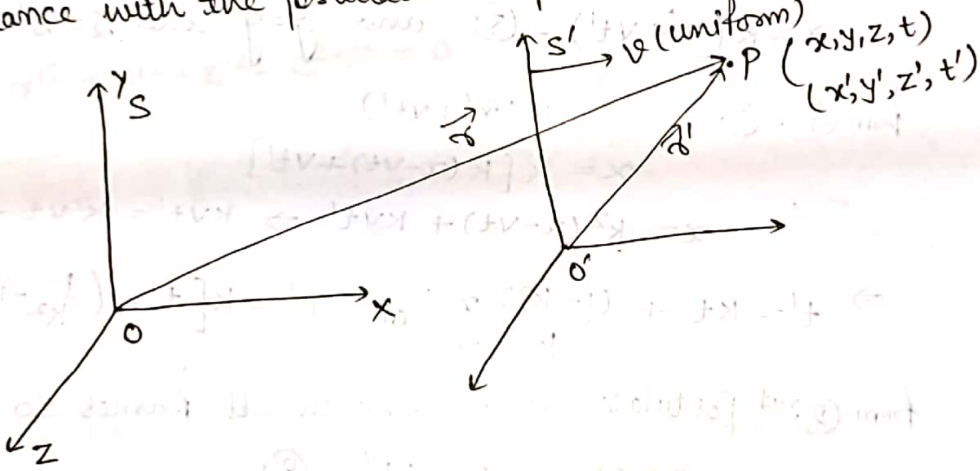
Fundamental Postulate of Einstein Theory of Relativity →

① The fundamental laws of Physics are of the same form in all inertial frames of reference.

② The speed of the light is same in all inertial frames of reference, regardless of the motion of the source relative to the observer.

For Einstein in all frames (space-time) co-ordinates are Relative or changable.

* Lorentz's Transformation → "These are the equations which enable us to find the relation between the space and the time co-ordinates of an event in two different inertial frames in uniform relative motion w.r.t each other, in accordance with the postulate of special Theory of Relativity."



Both are inertial frame.
Suppose $t = t' = 0$ at O, O', when they coincide then a flash of light is sent out from 'O' along x-axis in wave-front. The light wave will travel outward in all direction with speed 'c' and hence will be an expanding sphere, At any time 't', to the observer of frame 's', at any time t, the light wave will appear a sphere of Radius 'ct', ($c \to$ in all frame is constant).

whose equation $t = \frac{OP}{c} = \frac{(x^2+y^2+z^2)^{1/2}}{c}$

$$\Rightarrow x^2 + y^2 + z^2 = c^2t^2 \quad \text{— ①}$$

and $t' = \frac{O'P}{c} = \frac{(x'^2+y'^2+z'^2)^{1/2}}{c}$

$$\Rightarrow x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{— ②}$$

according to Gallilean Transformation →

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

So from eqⁿ ② $(x-vt)^2 + y^2 + z^2 = c^2t^2$

$$\Rightarrow \quad x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 \quad — ③$$

Here eqⁿ ③ not resembles with ① Here $(-2xvt + v^2t^2)$ is extra factor, So Gallilean transformation are not satisfied.

Now from the Property of Homogenity and Isotropy of the free space.

① Transformations should be linear and simple

② at $v \lll c$ these Transformation change in Gallilean Transformation.

$$x' = k(x - vt) \quad — ④$$

from ①st Postulate all laws in nature same so

$$x = k(x' + vt') \quad — ⑤ \quad \text{and} \quad y = y' \text{ and } z = z' \quad — ⑥$$

from ④ & ⑤ $\quad x = K(x' + vt')$

$$x = K[K(x - vt) + vt']$$

$$x = K^2(x - vt) + Kvt' \Rightarrow Kvt' = K^2vt + (1 - K^2)x$$

$$\Rightarrow t' = Kt + \frac{(1-K^2)}{K}\frac{x}{v} \quad \text{or} \quad t' = K\left[t + \left(\frac{1}{K^2} - 1\right)\frac{x}{v}\right] \quad — ⑦$$

from ②nd Postulate c is same in all frames so

$$x = ct, \quad x' = ct' \quad — ⑧$$

from ④, ⑦, ⑧ so

$$K(x - vt) = cK\left[t + \frac{x}{v}\left(\frac{1}{K^2} - 1\right)\right]$$

$$\text{or} \quad \cancel{K}x\left[1 - \frac{c}{v}\left(\frac{1}{K^2} - 1\right)\right] = cKt + Kvt = \cancel{K}t(c + v)$$

$$\cancel{ct}\left[\cancel{1} - \frac{c}{v}\left(\frac{1}{K^2} - 1\right)\right] = \cancel{ct}\left(\cancel{1} + \frac{v}{c}\right)$$

$$\frac{c}{v}\left(1 - \frac{1}{K^2}\right) = \frac{v}{c} \quad \Rightarrow \quad \frac{1}{K^2} = 1 - \frac{v^2}{c^2}$$

$$\Rightarrow \quad K = \frac{1}{\sqrt{1 - v^2/c^2}}$$ is never negative because $v \lll c$.

from ⑦ $\quad t' = K\left[t + \left(\cancel{1} - \frac{v^2}{c^2} - \cancel{1}\right)\frac{x}{v}\right] = K\left[t - \frac{vx}{c^2}\right]$

$$\Rightarrow \quad \boxed{t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}}$$

So Lorentz Transformation equations are →

Transform S to S' is

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Now take Lorentz Inverse Transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

If $v \ll c$ So $\frac{v^2}{c^2} \ll 1$, So they change in to Gallilean Transformation.

<u>Numerical</u> → Prove that the spherical wave-front of light is Invarient under Lorentz's Transformation.

or $x^2 + y^2 + z^2 - c^2t^2 = 0.$

$$\Rightarrow \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

$$\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right)^2 + y^2 + z^2 - c^2\left[\frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}\right]^2 = 0.$$

$$\frac{(x - vt)^2 - c^2\left(t - \frac{vx}{c^2}\right)^2}{1 - v^2/c^2} + y^2 + z^2 = 0.$$

$$\Rightarrow \quad \frac{x^2 + v^2t^2 - \cancel{2xvt} - c^2t^2 - \frac{v^2x^2}{c^2} + \cancel{2xvt}}{1 - v^2/c^2} + y^2 + z^2 = 0.$$

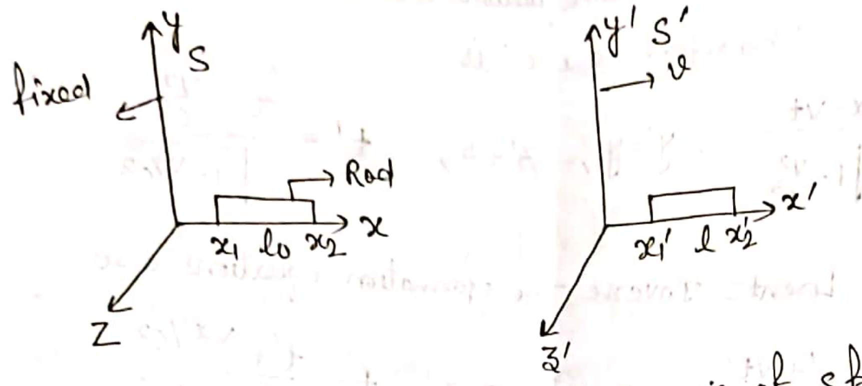
$$\Rightarrow \quad \frac{x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)}{1 - v^2/c^2} + y^2 + z^2 = 0.$$

$$\Rightarrow \quad x^2 - c^2t^2 + y^2 + z^2 = 0.$$ <u>Prove that.</u>

* Consequences of Lorentz Transformation →

1. Length Contraction or Lorentz-Fitzerland Contraction.
2. Time dialation
3. Transformation of velocities or Addition of velocities
4. Transformation of Acceleration
5. Relativity of Simultanity
6. Relativity of mass
7. Mass - Energy Equivalance.

① Length Contraction ⇒



Consider a Rod lying at rest along x-axis of S frame, so.

Proper length of Rod $l_0 = x_2 - x_1$

Now we see measurement in S' frame which are going in +ve direction from S with velocity 'v' → so

observe length $l = x_2' - x_1'$

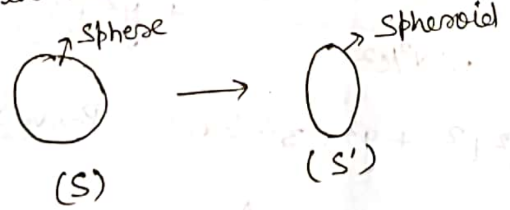
Here $x_1 = \dfrac{x_1' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$, $x_2 = \dfrac{x_2' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$

So $x_2 - x_1 = \dfrac{x_2' - x_1'}{\sqrt{1-v^2/c^2}}$ ⇒ $\boxed{l_0 = \gamma l}$ $l_0 > l$ $\left[\because \gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ or } \gamma > 1\right]$

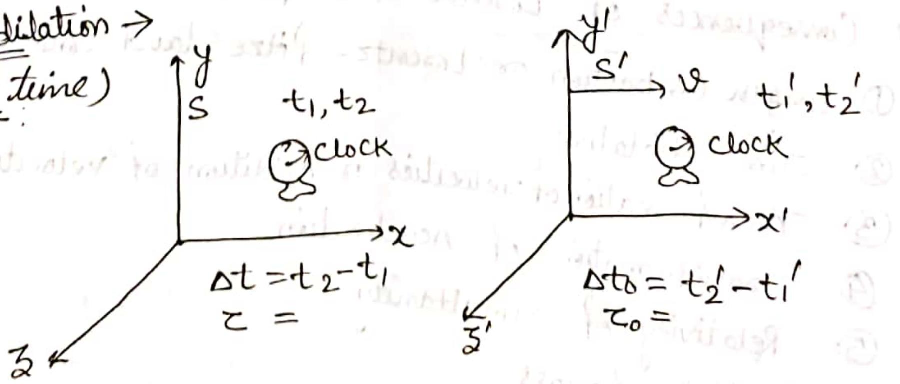
But in Perpendicular directions $y' = y$, $z' = z$ no change in length.

Thus length of the Rod in all other frame of Reference in uniform motion with respect to the frame in which the Rod is at Rest, is Shorter than its Proper length.

Example → A Sphere will look like as a spheroid due to decrease in its diameter Parallel to x-axis.



*② Time-dilation →
(Relativity of time)



Consider a clock placed at the point x' in the frame S' moving with uniform velocity 'v' along x-axis with respect to frame 'S'. Suppose at any instant, observer of frame S' for which clock is at rest, have time t_1' so the observer of frame 'S' will find the time to be

$$\boxed{t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

If some time later, the observer of frame S' notes the time as t_2', the observer of frame S will record it as

$$\boxed{t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-v^2/c^2}}}$$

So $t_2 - t_1 = \dfrac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}}$ $\Rightarrow$ $\tau = \dfrac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\tau_0$ and $\tau > \tau_0$

So time interval noted by an observer w.r.t. whom the clock is at rest is smaller than the time interval noted by the observer w.r.t whom clock is in motion.

* (4) Velocity Transformation (Relativistic Addition of velocities) →

Consider a body moving with a constant linear velocity u w.r.t. s frame along x-axis and u' w.r.t s' frame along x'-axis. frame s' moves with velocity v in same direction w.r.t frame s.

Suppose u_x, u_y, u_z are component of velocity w.r.t frame s and u_x', u_y', u_z' are component of velocity w.r.t frame s'.

Here s frame fixed.

In s frame $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, $u_z = \frac{dz}{dt}$

In s' frame $u_x' = \frac{dx'}{dt}$, $u_y' = \frac{dy'}{dt}$; $u_z' = \frac{dz'}{dt}$

According to lorentz Transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Now differentiate it

$$dx' = \frac{dx - Vdt}{\sqrt{1-V^2/c^2}}, \quad dy' = dy, \quad dz' = dz, \quad dt' = \frac{dt - \frac{Vdx}{c^2}}{\sqrt{1-V^2/c^2}}$$

Now $$u_x' = \frac{dx'}{dt'} = \frac{dx - Vdt}{dt - \frac{Vdx}{c^2}} = \frac{\frac{dx}{dt} - V}{1 - \frac{V}{c^2}\frac{dx}{dt}} = \frac{u_x - V}{1 - \frac{V}{c^2}u_x} \quad \text{— (1)}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy\sqrt{1-V^2/c^2}}{dt - \frac{Vdx}{c^2}} = \frac{dy/dt\sqrt{1-V^2/c^2}}{1 - \frac{V}{c^2}\frac{dx}{dt}}$$

$$u_y' = \frac{u_y\sqrt{1-V^2/c^2}}{1 - \frac{V}{c^2}u_x} \quad \text{— (2)}$$

Similarly $$u_z' = \frac{u_z\sqrt{1-V^2/c^2}}{1 - \frac{Vu_x}{c^2}} \quad \text{— (3)}$$

So equations (1) (2) (3) give Transformation equations for velocity Components in S to S' frame.

→ The inverse velocity Transformation equations from S' to S frame is

$$u_x = \frac{u_x' + V}{1 + \frac{Vu_x'}{c^2}}, \quad u_y = \frac{u_y'\sqrt{1-V^2/c^2}}{1 + \frac{V}{c^2}u_x'}, \quad u_z = \frac{u_z'\sqrt{1-V^2/c^2}}{1 + \frac{Vu_x'}{c^2}}$$

Case (1) when $V \ll c$ so eqn (1) (2) (3) are

$$u_x' = u_x - V, \quad u_y' = u_y, \quad u_z' = u_z.$$

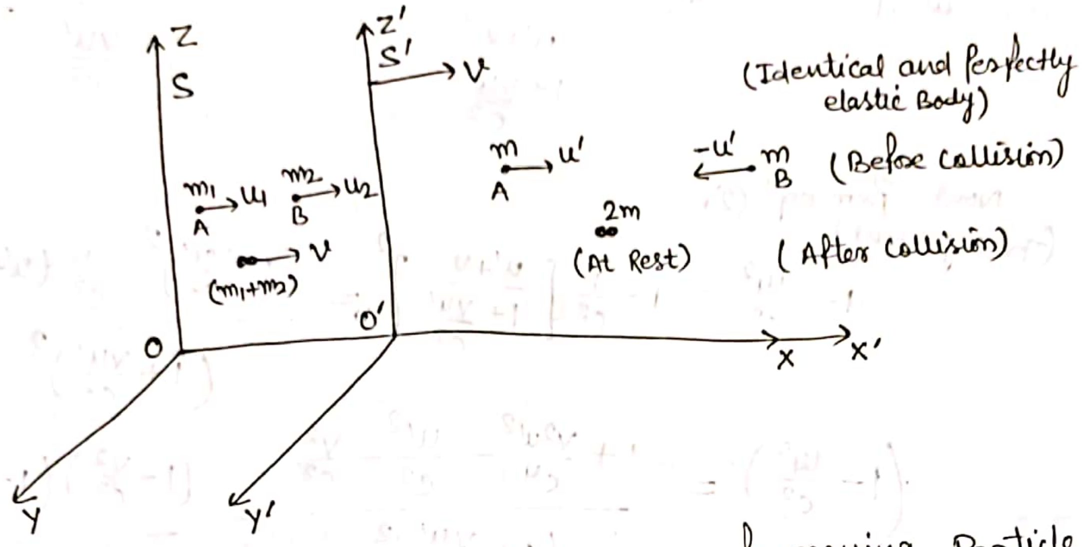
Called Classical (Newtonian) Gallilean Law of addition of velocity.

Case (2) If we consider the Particle to be a photon moving with velocity 'c' in frame S' which is also moving with velocity 'c' along x-axis so $$u_x = \frac{u_x' + V}{1 + \frac{Vu_x'}{c^2}}$$

If $u_x' = c$ $$u_x = \frac{c+V}{1+\frac{Vc}{c^2}} = \frac{c+V}{1+\frac{V}{c}} = c$$

So speed of light is an absolute Constant, independent of the motion of the frame of reference and all frame of Reference.

⑤ Relativity of mass (variation of mass with velocity) ⇒



According to newtonic (Classical) mechanics mass of moving particle does not depend on velocity. But Relatively see below →.

✓ Suppose to the observer of frame S, the masses of the bodies 'A' and 'B' appears to be m_1 and m_2 and velocities u_1 and u_2 after collision the two bodies come to rest momentarily in frame S' they together will appear to be moving with the velocity of frame S' with velocity v to the observer of frame S.

According to law of conservation of momentum is →

$$m_1u_1 + m_2u_2 = (m_1+m_2)v \quad (*)$$

$$\frac{m_1}{m_2} = \left(\frac{v-u_2}{u_1-v}\right) \quad -\ ①$$

from inverse velocity Transformation equation we have

$$u_x = \frac{u_x'+v}{1+\frac{vu_x'}{c^2}}$$

The Body A moves with velocity u' in frame S' and appears to be moving with velocity u_1 to the observer of frame S.

Set $u_x' = u'$ and $u_x = u_1$

So velocity of Body A in frame S is $u_1 = \dfrac{u'+v}{1+\frac{vu'}{c^2}} \quad -\ ②$

Similarly set $u_x' = -u'$ and $u_x = u_2$

So velocity of Body 'B' in frame S is $u_2 = \dfrac{-u'+v}{1-\frac{vu'}{c^2}} \quad -\ ③$

from eq^n ② & ③ in ①

$$\frac{m_1}{m_2} = \frac{v - \left(\frac{-u'+v}{1-\frac{vu'}{c^2}}\right)}{\frac{u'+v}{1+\frac{vu'}{c^2}} - v} = \frac{1+\frac{vu'}{c^2}}{1-\frac{vu'}{c^2}} \quad — ④$$

Now from eq^n ②
(Tricky Point)

$$1-\frac{u_1^2}{c^2} = 1-\frac{1}{c^2}\left[\frac{u'+v}{1+\frac{vu'}{c^2}}\right]^2 = \frac{\left(1+\frac{vu'}{c^2}\right)^2 - \frac{1}{c^2}(u'+v)^2}{\left(1+\frac{vu'}{c^2}\right)^2}$$

$$\left(1-\frac{u_1^2}{c^2}\right) = \frac{1+\frac{v^2u'^2}{c^4} - \frac{u'^2}{c^2} - \frac{v^2}{c^2}}{\left(1+\frac{vu'}{c^2}\right)^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{vu'}{c^2}\right)^2}$$

$$\Rightarrow \left(1+\frac{vu'}{c^2}\right)^2 = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_1^2}{c^2}\right)}$$

$$\Rightarrow \left(1+\frac{vu'}{c^2}\right) = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_1^2}{c^2}\right)}} \quad — ⑤$$

Similarly from eq^n ⑤ $u' \to -u'$ & $(u_1 \Leftrightarrow u_2)$

$$\left(1-\frac{vu'}{c^2}\right) = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_2^2}{c^2}\right)}} \quad — ⑥$$

eq^n ④ & ⑤ Put in ④

$$\frac{m_1}{m_2} = \frac{\sqrt{1-u_2^2/c^2}}{\sqrt{1-u_1^2/c^2}}$$

Now if w.r.t S frame, Before Collision velocity of Particle B is zero

So $u_2 = 0$.

$$\frac{m_1}{m_2} = \frac{1}{\sqrt{1-u_1^2/c^2}} \quad \text{or} \quad m_1 = \frac{m_2}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

So Body 'B' at Rest so $m_2 = m_0$

$$\boxed{m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}}$$

Hence above equation can be considered to be applicable to a Single body whose rest mass is m_0 and moves with a velocity v so

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

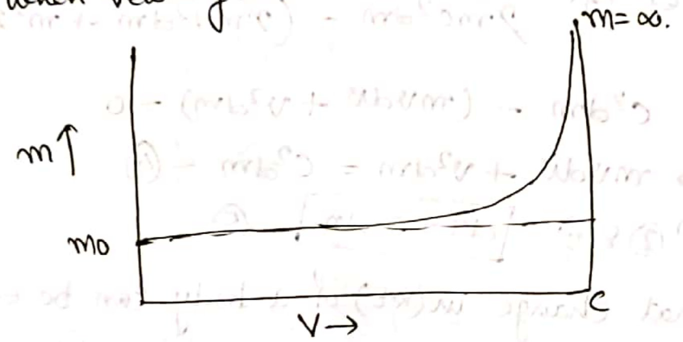
✓ Discussion of Result →

① when $v \ll c$ So $m = m_0$ (likely Classical Mechanics)

② If $v = c$ So $m = \infty$ (which is impossible)

If $v > c$ So $m =$ imaginary (which is impossible)

also say when velocity increase so effective mass of body increased.



✓ Experimental verification →

① for High energy electrons and β-Particles emitted by some Radio-active Substance by Bucherer, Kauffmann, Guge and Lavanchy)

② splitting of spectral line in H-spectrum. and Phenomenon of fine-structure of H-spectrum By Sommerfeld Relativistic Co-rrection.

③ Particle accelerator (cyclotron, Betatron) have mass increase with velocity increase.

* Mass- Energy Equivalance →

Mass is depend on velocity So K.E is also change with velocity. use Newton Second Law & work energy Theorem both are invarient in all frame by 1st Postulate.

✓ Suppose a force 'F' act over a body whose Rest mass is 'm_0' over a distance dx, The amount of work-done by the force will appear as increase in K.E (dT) So

$$dT = F\,dx \quad \text{—— ①}$$

we know $F = \frac{d\vec{P}}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$

Here m & v are variable and m_0 & c are Constant Quantity.

So $dT = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$

$dT = mv\,dv + v^2 dm$ — ② $\left(\because v = \frac{dx}{dt} \text{ at any instant}\right)$

we know $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ or $m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}}$

$\Rightarrow m^2 = \frac{m_0^2c^2}{c^2-v^2} \Rightarrow m^2c^2 - m^2v^2 = m_0^2c^2$ — ③

differentiate eq^n ③ $\cancel{2}\cancel{m}c^2dm - (\cancel{2}\cancel{m}v^2dm + m^{\cancel{2}}\cancel{2}v\,dv) = 0$

$c^2dm - (mv\,dv + v^2dm) = 0$

$\Rightarrow mv\,dv + v^2dm = c^2dm$ — ④

from eq^n ② & ④ $\boxed{dT = c^2dm}$ — ⑤

It shows that change in (K.E) of a body can be expressed in terms of change in its mass due to motion.

✓ when a Body is accelerated from Rest to a velocity 'v', it's mass increases from m_0 to m and K.E acquired is obtained by integrating eq^n ⑤ between the limits m_0 to m Therefore →

$$T = \int_{m_0}^{m} c^2dm = c^2(m - m_0)$$

✓ So K.E energy of moving Particle is equal to c^2 times the gain in mass due to motion.

✓ m_0 is Rest mass of the Particle and m_0c^2 is Rest mass Energy Called Internal Energy.

So Total Energy $E = T + m_0c^2 = c^2(m - m_0) + m_0c^2 = mc^2$ — ⑥

So $E = mc^2$ is Einestein mass- Energy Equivalance theorem.

✓ Discussion of the Result →

① Relation $E = mc^2$ Shows that equivalance of mass and energy So Thus Special Theory of Relativity ascribes energies to all masses and masses to all energies.

(2) In classical Mechanics, the Law of Conservation of mass and energy are two seperate Principles independent of each other. The Relations $E = mc^2$ leads to unification of the two laws into one law called law of Conservation of Relativistic energy

(3) In classical Mechanics mass is considered something fundamental to matter while energy is a Property of the matter acquired by virtue of its Position or motion. The Relation $E = mc^2$ Puts an end to such a distinction between mass and Energy.

(4) The kinetic energy of a Particle travelling with a velocity v is

$$T = c^2 (m - m_0)$$

Here $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

So $T = c^2 \left[m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - m_0 \right]$

$$T = m_0 c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right]$$

$$T = m_0 c^2 \left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \right) - 1 \right] \quad \left[\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\lfloor 2} x^2 \right]$$

$$T = \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \cdots$$

when $v \ll c$ so.

$\boxed{T \simeq \frac{1}{2} m_0 v^2}$ → formula of K.E in classical Picture.

* Experimental Evidence in Support of the mass-Energy Equivalance →

(1) for electron $m_0 = 9.1 \times 10^{-31}\,kg$

So $E = mc^2 = \frac{9.1 \times 10^{-31} \times 9 \times 10^{16}}{1.6 \times 10^{-13}}\,MeV = .511\,MeV$

$[\because 1eV = 1.6 \times 10^{-19} J]$

for 1 amu $= 1.67 \times 10^{-27}\,kg$

So $\boxed{1\ amu = 931\ MeV}$

(2) Pair-Production and Anhillation of matter also support the equivalance of mass and Energy.

(3) Fission and Fusion Processes are the direct applications of the Einestein's mass-Energy Relation.